Unit II-LaplaceTransform

LAPLACE TRANSFORM

Introduction : Laplace Transformation named after a Great French mathematician PIERRE SIMON DE LAPLACE (1749-1827) who used such transformations in his researches related to "Theory of Probability".

The powerful practical Laplace transformation techniques were developed over a century later by the English electrical Engineer OLIVER HEAVISIDE (1850-1925) and were often called "Heaviside -Calculus".

1 Laplace Transform :

Definitions

1. Transformation:

A "Transformation" is an operation which converts a mathematical expression to a different but equivalent form

2. Laplace Transformation :

Let a function f(t) be continuous and defined for positive values of 't'. The Laplace transformation of f(t) associates a function s defined by the equation

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt, \quad t > 0$$

Here, F(s) is said to be the Laplace transform of f(t) and it is written as L[f(t)] or L[f]. Thus F(s) = L(f(t))

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt, \quad t > 0$$

3. Exponential Order : A function f(t) is said to be of exponential order if

$$\underset{t\to\infty}{Lt}\, e^{-st} f(t) = 0$$

1.1Laplace Transform – Sufficient Conditions For Existence

i) f(t) should be continuous or piecewise continuous in the given closed interval [a, b] where a > 0.

ii) f(t) should be of exponential order.

Example:

1. L[tan t] does not exist since tan t is not piecewise continuous. i.e., tan t has infinite number of infinite

$$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

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1.2 Problems Based On Laplace Transform – Sufficient Conditions For Existence

1. Show that t^2 is of exponential order.

Solution:

Given $f(t) = t^2$

By the definition of exponential order,

$$\underset{t\to\infty}{Lt}\, e^{-st} f(t) = \underset{t\to\infty}{Lt}\, e^{-st} t^2$$

$$= \underset{t\to\infty}{Lt} \frac{t^2}{e^{st}} \left[\frac{\infty}{\infty} \text{ i.e., Indeterminant form}\right]\left[\text{Apply L'Hospital rule}\right]$$

$$= \underset{t\to\infty}{Lt} \frac{2t}{se^{st}} \left[\frac{\infty}{\infty} \text{ i.e., Indeterminant form}\right]\left[\text{Apply L'Hospital rule}\right]$$

$$= \underset{t\to\infty}{Lt} \frac{2}{s^2 e^{st}}$$

$$= \frac{2}{\infty}$$

$$\underset{t\to\infty}{Lt}\, e^{-st} t^2 = 0$$

Hence t2 is of exponential order.

2. Show that the function the following function is not of exponential order $f(t) = e^{t^2}$

Solution:

Given $f(t) = e^{t^2}$

By the definition of exponential order,

$$\underset{n\to\infty}{Lt}\, e^{-st} e^{t^2} = \underset{n\to\infty}{Lt}\, e^{-st+t^2}$$

$$= e^{\infty}$$

$$\underset{n\to\infty}{Lt}\, e^{-st} e^{t^2} = \infty$$

1.3 Define function of class A :

A function which is sectionally continuous over any finite interval and is of exponential order is known as a function of class A

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2 Transforms Of Elementary Functions- Basic Properties

Important Results

1. $L[1] = \frac{1}{S}$ where $s > 0$

2. $L[t^n] = \frac{n!}{s^{n+1}}$ where n= 0,1,2,3,...........

3. $L[t^n] = \frac{\Gamma n+1}{s^{n+1}}$ where n is not a integer

4. $L[e^{at}] = \frac{1}{s-a}$ where $s > a$ or $s - a > 0$

5. $L[e^{-at}] = \frac{1}{s+a}$ where $s + a > 0$

6. $L[\sin at] = \frac{a}{s^2+a^2}$ where $s > 0$

7. $L[\cos at] = \frac{s}{s^2+a^2}$ where $s > 0$

8. $L[\sinh at] = \frac{a}{s^2-a^2}$ where $s > |a|$ or $s^2 > a^2$

9. $L[\cosh at] = \frac{s}{s^2 - a^2}$ where $s^2 > a^2$

10. *Linearity property*

$$L[af(t) \pm bg(t)] = a\,L[f(t)] \pm bL[g(t)]$$

2.1 Problems Based On Transforms Of Elementary Functions- Basic Properties

1. Find $L[t^5 + e^{3t} + 5e^{-2t}]$

Solution:

$$L[t^5 + e^{3t} + e^{-2t}] = L[t^5] + L[e^{3t}] + L[5e^{-2t}]$$

$$= \frac{5!}{s^{5+1}} + \frac{1}{s-3} + \frac{5}{s+2}$$

$$\therefore L[t^5 + e^{3t} + e^{-2t}] = \frac{5!}{s^6} + \frac{1}{s-3} + \frac{5}{s+2}$$

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2. Find L[sin 2t+ cos πt – 8cosh 7t +sinhbt]

Solution:

$$L[\sin 2t + \cos \pi t - 8\cosh 7t + \sinh bt]$$
$$= \frac{2}{s^2+2^2} + \frac{s}{s^2+\pi^2} - \frac{8s}{s^2-7^2} + \frac{b}{s^2-b^2}$$
$$= \frac{2}{s^2+4} + \frac{s}{s^2+\pi^2} - \frac{8s}{s^2-49} + \frac{b}{s^2-b^2}$$

3. *Find* $L\left[\frac{1}{\sqrt{t}}\right]$

Solution:

$$Given\ L\left[\frac{1}{\sqrt{t}}\right] = L\left[t^{-1/2}\right]$$
$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}$$
$$= \frac{\Gamma \frac{1}{2}}{s^{\frac{1}{2}}}$$
$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$\text{Hence } L\left[\frac{1}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}$$

2.2 First Shifting Theorem

$$If\ L[f(t)] = F(s), \text{ then } L[e^{at} f(t)] = F(s-a)$$

$$If\ L[f(t)] = F(s), \text{ then } L[e^{-at} f(t)] = F(s+a)$$

2.3 Second Shifting Theorem

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$$\text{If } L\left[f(t)\right] = F(s) \text{ and } G(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

$$\text{then } L[G(t)] = e^{-as}F(s)$$

2.4 Problems Based On First And Second Shifting Theorem

1. Find $L[t^n e^{-at}]$

Solution:

$$L[t^n e^{-at}] = \left[L[t^n]\right]_{s\to(s+a)}$$

$$= \left[\frac{n!}{s^{n+1}}\right]_{s\to(s+a)}$$

$$\therefore L[t^n e^{-at}] = \left[\frac{n!}{(s+a)^{n+1}}\right]$$

2. Find $L[e^{at} \sinh bt]$

Solution

$$L[e^{at}\sinh bt] = \left[L[\sinh bt]\right]_{s\to(s-a)}$$

$$= \left[\frac{b}{s^2 - b^2}\right]_{s\to(s-a)}$$

$$\therefore L[e^{at}\sinh bt] = \frac{b}{(s-a)^2 - b^2}$$

2.5 Tutorial Problems:

1. Find L[cos 4t sin 2t]
2. Find $L[\sinh^2 2t]$
3. Find L[cos(3t-4)]
4. Find $L[e^{-t}t^9]$

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3 Transforms Of Derivatives And Integrals Of Functions
Properties:

$L[f'(t)] = s\,L[f(t)] - f(0)$
$L[f''(t)] = s^2\,L[f(t)] - s f(0) - f'(0)$

3.1 Transform of integrals

$$\text{If } L[f(t)] = F(s), \text{ then } L\left[\int_0^t f(u)\,du\right] = \frac{1}{s} L[f(t)]$$

3.2 Derivatives of transform

$$\textit{If } L[f(t)] = F(s) \text{ then } L[t\,f(t)] = -\frac{d}{ds}F(s) = -F'(s)$$
$$\textit{If } L[f(t)] = F(s) \text{ then } L[t^n f(t)] = (-1)^n F^{(n)}(s)$$

3.3 Problems Based On Derivatives Of Transform

1. Find L[t sin at]
Solution:
We know that

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$$If\ L[f(t)] = F(s) \text{ then } L[t\,f(t)] = -\frac{d}{ds}F(s) = -F'(s)$$

$$L[t\sin at] = -\frac{d}{ds}L[\sin at]$$

$$= -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]$$

$$= -\left(\frac{(s^2+a^2)(0)-a(2s)}{\left(s^2+a^2\right)^2}\right)$$

$$= -\left(\frac{-a(2s)}{\left(s^2+a^2\right)^2}\right)$$

$$\therefore L[t\sin at] = \frac{2as}{\left(s^2+a^2\right)^2}$$

2. *Show that* $\int_0^\infty e^{-t}\,t\cos t\,dt = 0$

Solution:

$$Given\ \int_0^\infty e^{-t}\,t\cos t\,dt = [L[t\cos t]]_{s=1}$$

$$= \left[-\frac{d}{ds}L(\cos t)\right]_{s=1}$$

$$= \left[-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s=1}$$

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$$=\left[-\left[\frac{(s^2+1)(1)-s(2s)}{\left(s^2+1\right)^2}\right]\right]_{s=1}$$

$$=\left[-\left[\frac{s^2+1-2s^2}{\left(s^2+1\right)^2}\right]\right]_{s=1}=\left[-\left[\frac{1-s^2}{\left(s^2+1\right)^2}\right]\right]_{s=1}=[-0]$$

$$\therefore \int_0^\infty e^{-t}\, t\cos t\, dt = 0$$

2. *Find* $L\left[\frac{\cos at-\cos bt}{t}\right]$

Solution:

$$\text{Given } L\left[\frac{\cos at-\cos bt}{t}\right]=\int_s^\infty L[\cos at-\cos bt]\,ds$$

$$=\int_s^\infty\left[\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right]ds$$

$$=\frac{1}{2}\left[\log\left(s^2+a^2\right)-\log\left(s^2+b^2\right)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$=\frac{1}{2}\left[0-\log\frac{s^2+a^2}{s^2+b^2}\right] \qquad (\because \log 1=0)$$

$$\therefore L\left[\frac{\cos at-\cos bt}{t}\right]=\frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]$$

3.4 Tutorial Problems:

1. Find L[t sin2t]
2. Find L[t cos at]
3. Find L[t sin3t cos2t]

3.5 Problems Based On Integrals Of Transform

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1. *Find* $L\left[\frac{\sin 3t \cos t}{t}\right]$

Solution:

Given $L\left[\frac{\sin 3t \cos t}{t}\right] = L\left[\frac{\sin 4t + \sin 2t}{2t}\right]$

$$= \frac{1}{2} L\left[\frac{\sin 4t + \sin 2t}{t}\right]$$

$$= \frac{1}{2}\left[L\left[\frac{\sin 4t}{t}\right] + L\left[\frac{\sin 2t}{t}\right]\right]$$

$$= \frac{1}{2}\left(\cot^{-1}\left(\frac{s}{4}\right) + \cot^{-1}\left(\frac{s}{2}\right)\right) \qquad \left(\because L\left[\frac{\sin at}{t}\right] = \cot^{-1}\left(\frac{s}{a}\right)\right)$$

$$\therefore L\left[\frac{\sin 3t \cos t}{t}\right] = \frac{1}{2}\left(\cot^{-1}\left(\frac{s}{4}\right) + \cot^{-1}\left(\frac{s}{2}\right)\right)$$

3.6 Tutorial Problems:

1. *Find* $L\left[e^t(\cosh 2t + \frac{1}{2}\sinh 2t\right]$

2. *Using Laplace transform prove that* $\int_0^\infty \frac{1-\cos 2t}{t^2}\,dt = \pi$

4 Transforms Of Unit Step Function And Impulse Function

4.1 Problems Based On Unit Step Function (Or) Heaviside's Unit Step Function

1. Define the unit step function.

Solution:

The unit step function, also called Heaviside's unit function is defined as

$$U(t-a) = \begin{cases} 0 & for\ t < a \\ 1 & for\ t > a \end{cases}$$

This is the unit step functions at t = a. It can be also denoted by H(t-a).

2. Give the L.T of the unit step function.

Solution:

The L.T. of the unit step function is given by

$$L[U(t-a)] = \int_0^\infty e^{-st}\ U(\mathrm{t}-\mathrm{a})\,\mathrm{dt}$$

$$= \int_0^a e^{-st}(0)\,\mathrm{dt} + \int_a^\infty e^{-st}(1)\,\mathrm{dt}$$

$$= \int_a^\infty e^{-st}\,\mathrm{dt}$$

$$= \left[\frac{e^{-st}}{-s}\right]_s^\infty$$

$$= 0 - \left[\frac{e^{-sa}}{-s}\right]$$

$$\therefore L[U(t-a)] = \left[\frac{e^{-as}}{s}\right]$$

4.2 Tutorial Problems:

1. Give the L.T. of the Dirac Delta function
2. Find the L.T. of t U(t-9).
3. Find $L^{-1}[1]$

5 Transform Of Periodic Functions

Definition: (Periodic)

A function f(x) is said to be "periodic" if and only if f(x+p) = f(x) is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The Laplace Transformation of a periodic function f(t) with period p given by

$$\frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(\mathrm{t})\,\mathrm{dt}$$

5.1 Problems Based On Transform Of Periodic Functions

1. Find the Laplace Transform of the Half-sine wave rectifier function

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$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

Solution:

We know that

$$L[f(t)] = \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\, dt$$

$$L[\sin \omega t] = \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \left(\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\, dt + 0 \right)$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \left(\frac{e^{-st}}{s^2+\omega^2} [-s \sin \omega t - \omega \cos \omega t] \right)_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \left(\frac{e^{-s\pi/\omega}\omega + \omega}{s^2+\omega^2} \right)$$

$$= \frac{\omega\left(1+e^{-s\pi/\omega}\right)}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)(s^2+\omega^2)}$$

$$= \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)(s^2+\omega^2)}$$

$$\therefore L[\sin \omega t] = \frac{\omega}{(s^2+\omega^2)\left(1-e^{\frac{-\pi s}{\omega}}\right)}$$

5.2 Tutorial Problems:

1. Find the Laplace Transform of triangular wave function

$$f(t) = \begin{cases} t & , 0 < t < a \\ 2a - t, & a < t < 2a \text{ with } f(t+2a) = f(t) \end{cases}$$

2. Find the Laplace transform of the square wave function (Meoander function) of period 'a' defined as

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$$f(t) = \begin{cases} 1\ , & 0 < t \le \frac{a}{2} \\ -1, & \frac{a}{2} < t < a \end{cases}$$

6 Inverse Laplace Transform

a.If L[f(t)] = F(s), then L−1[F(s)] = f(t) where L−1 is called the inverse Laplace transform operator.

b.If F1(s) and F2(s) are L.T. of f(t) and g(t) respectively then

$$L^{-1}[C_1 F_1(s) + C_2 F_2(s)] = C_1 L^{-1}[F_1(s)] + C_2 L^{-1}[F_2(s)]$$

Important Formulas

1. $L^{-1}\left[\frac{1}{s}\right] = 1$
2. $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\lfloor n-1}$
3. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
4. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$
5. $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$
6. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

7. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

8. $L^{-1}[F(s-a)] = e^{at}\ f(\mathrm{t})$

9. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin bt$

10. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt$

11. $L^{-1}\left[\frac{1}{(s-a)^2-b^2}\right] = \frac{1}{b}e^{at}\sinh bt$

12. $L^{-1}\left[\frac{s-a}{(s-a)^2-b^2}\right] = e^{at}\cosh bt$

13. $L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right] = \frac{1}{2a}\mathrm{t}\ \sin at$

14. $L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)^2}\right] = \frac{1}{2a}[\sin at + at\cos at]$

15. $L^{-1}\left[\frac{1}{\left(s^2+a^2\right)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$

16. $L^{-1}\left[\frac{s^2-a^2}{\left(s^2+a^2\right)^2}\right] = \mathrm{t}\ \cos at$

17. $L^{-1}[1] = \delta(\mathrm{t})$

6.1 Problems based on Inverse Laplace Transform

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1. *Find* $L^{-1}\left[\frac{2s}{s^2-16}\right]$

Solution:

Given

$$L^{-1}\left[\frac{2s}{s^2-16}\right] = 2L^{-1}\left[\frac{s}{s^2-16}\right]$$
$$= 2\cosh 4t$$

2. Find

$$L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]$$

Solution:

$$L^{-1}\left[\frac{s-3}{s^2+4s+13}\right] = L^{-1}\left[\frac{s-3}{(s+2)^2+13-4}\right]$$
$$= L^{-1}\left[\frac{s-3}{(s+2)^2+9}\right]$$
$$= L^{-1}\left[\frac{s+2-5}{(s+2)^2+9}\right]$$
$$= L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] - 5L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$$
$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] - \frac{5}{3}L^{-1}\left[\frac{3}{(s+2)^2+3^2}\right]$$
$$= e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}L^{-1}\left[\frac{3}{s^2+3^2}\right]$$

$$\therefore L^{-1}\left[\frac{s-3}{s^2+4s+13}\right] = e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}\sin 3t$$

6.2 Inverse Laplace Transforms of derivatives of F(s)

Unit II-LaplaceTransform

$$If \quad L^{-1}[F(s)] = f(t), \text{then } L^{-1}[F'(s)] = -t\, f(t)$$
$$= -t\, L^{-1}[F(s)]$$

6.3Problems based on Inverse Laplace Transforms of derivatives of F(s)

1. *Find* $L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right]$

Solution:

$$Let \; F'(s) = \left[\frac{s}{(s^2-a^2)^2}\right]$$
$$\int F'(s)\,ds = \int\left[\frac{s}{(s^2-a^2)^2}\right]ds$$
$$F(s) = \int\left[\frac{s}{(s^2-a^2)^2}\right]ds$$
$$Put \; s^2-a^2 = t$$
$$2s\,ds = dt$$
$$sds = \frac{dt}{2}$$

$$= \int \frac{1}{t^2}\frac{dt}{2} = \frac{1}{2}\left[\frac{-1}{t}\right]$$
$$= -\frac{1}{2t}$$
$$\therefore F(s) = -\frac{1}{2(s^2-a^2)}$$

6.4 Inverse Laplace Transform of Integrals

$$L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{1}{t}f(t) = \frac{1}{t}L^{-1}[F(s)]$$

(or)

$$L^{-1}[F(s)] = t\,L^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

6.5 Problems based on Inverse Laplace Transform of Integrals

1. *Find* $L^{-1}\left[\frac{2s}{(s^2-1)^2}\right]$

Solution:

We know that

$$L^{-1}[F(s)] = t\,L^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

$$L^{-1}\left[\frac{2s}{(s^2-1)^2}\right] = t\;L^{-1}\left[\int_s^\infty \frac{2s}{(s^2-1)^2}\,ds\right]$$

$$= t\;L^{-1}\left[\left(\frac{-1}{(s^2-1)}\right)_s^\infty\right]$$

$$= t\;L^{-1}\left[0+\frac{1}{s^2-1}\right]$$

$$= t\;L^{-1}\left[\frac{1}{s^2-1}\right]$$

$$\therefore L^{-1}\left[\frac{2s}{(s^2-1)^2}\right] = t\,\sinh t$$

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6.6 Problems based on Partial fractions method

1. *Find* $L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]$

Solution:

Consider

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{A}{s+1}+\frac{B}{S-2}+\frac{C}{(S-2)^2}+\frac{D}{(S-2)^3}$$

$$5s^2-15s-11=A(S-2)^3+B(S+1)(S-2)^2+C(S+1)(S-2)+D(S+1)$$

Put s = -1, we get

$$5+15+-11=A(-1-2)^3$$

$$9=-27A$$

$$A=\frac{-1}{3}$$

Equating the coefficients of s^3 on both sides, we get

$$0 = A+B$$

$$B=-A$$

$$B=\frac{1}{3}$$

Put s = 2, we get

$$-21=D$$

$$D=-7$$

Put s = 0, we get

$$-11 = -8A + 4B - 2C + D$$

$$= -8\left(\frac{-1}{3}\right) + 4\left(\frac{1}{3}\right) - 2C - 7$$

$$-4 = \frac{8}{3} + \frac{4}{3} - 2C$$

$$-8 = -2C$$

$$C = 4$$

$$\therefore \frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} = \frac{\frac{-1}{3}}{s+1} + \frac{\frac{1}{3}}{S-2} + \frac{4}{(S-2)^2} - \frac{7}{(S-2)^3}$$

$$L^{-1}\left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right] = \frac{-1}{3}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right] - 7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$= \frac{-1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right] - 7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$= \frac{-1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}\,t - \frac{7}{2}e^{2t}L^{-1}\left[\frac{2}{s^3}\right]$$

$$\therefore L^{-1}\left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right] = \frac{-1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}\,t - \frac{7}{2}e^{2t}t^2$$

6.7 Second Shifting property

$$L^{-1}[\mathrm{e}^{-as} F(\mathrm{s})] = f(t-a)U(t-a)$$

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6.8 Problems based on second shifting property

1. *Find* $L^{-1}\left[\frac{e^{-\pi s}}{s+3}\right]$

Solution:

Consider

$$L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$$

$$L^{-1}\left[\frac{e^{-\pi s}}{s+3}\right] = e^{-3(t-\pi)}U(\mathrm{t}-\pi)$$

6.9 Tutorial Problems:

a. Find the inverse L.T of Derivatives.

1. $\log\left(1+\frac{1}{s^2}\right)$

2. $\tan^{-1}(s+1)$

b. Partial Fraction Method

1. $\frac{s}{s^4+4a^4}$

2. $\frac{3s+1}{(\mathrm{s}-1)(\mathrm{s}^2+1)}$

6.10 Change of scale property

$$\text{If } L[f(\mathrm{t})]=\mathrm{F(s)}, \text{then } \mathrm{L}[f(at) = \frac{1}{a}F\left[\frac{s}{a}\right]$$

$$\text{If } f(\mathrm{t}) = \mathrm{L}^{-1}[F(\mathrm{s})], \text{ then } \mathrm{L}^{-1}[F(cs)] = \frac{1}{c}f\left[\frac{t}{c}\right]$$

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6.11 Problems based on Change of scale property

1. If $L[f(t)] = F(s)$ find $L\left[f\left(\frac{t}{a}\right)\right]$

Solution:

We know that
$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\left[f\left(\frac{t}{a}\right)\right] = \int_0^\infty e^{-st} f\left(\frac{t}{a}\right)dt$$

Put $u = \frac{t}{a}$ $\quad$ as $t \to 0 \Rightarrow u \to 0$

$du = \frac{dt}{a}$ $\quad$ $t \to \infty \Rightarrow u \to \infty$

$$L\left[f\left(t/a\right)\right] = \int_0^\infty e^{-s(au)} f(u)\,a\,du$$

$$= a\int_0^\infty e^{-sau} f(u)\,du$$

$$= a\int_0^\infty e^{-sat} f(t)\,dt$$

$$= a\,F[as]$$

6.12 Tutorial Problems:

1. If L[f (t)] = F(s) then L[F(t/2) = 2 F(2s)

2. Find $L^{-1}\left[\frac{s}{s^2a^2+b^2}\right]$

7 Convolution Theorem

If f (t) and g(t) are functions defined for t > = 0,
then L[f (t) * g (t)] = L[f (t)] L[-g (t)]

7.1 Problems on Convolution Theorem

1.Define convolution

Unit II-LaplaceTransform

The convolution of two functions f(t) and g(t) is defined as

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

Note: Convolution Integral or Falting integral

2. *Using convolution theorem find* $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$

Solution:

We know that $L^{-1}[F(s)\cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$

$$L^{-1}\left[\frac{1}{s+a}\cdot\frac{1}{s+b}\right] = L^{-1}\left[\frac{1}{s+a}\right] * L^{-1}\left[\frac{1}{s+b}\right]$$

$$= e^{-at} * e^{-bt}$$

Here $f(t) = e^{-at}$

$g(t) = e^{-bt}$

Unit II-LaplaceTransform

$$f(t)*g(t) = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t e^{-au}\, e^{-b(t-u)}\, du$$

$$= \int_0^t e^{-au}\, e^{-bt}\, e^{bu}\, du$$

$$= e^{-bt}\int_0^t e^{-(a-b)u}\, du$$

$$= e^{-bt}\left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t$$

$$= e^{-bt}\left[\frac{e^{-(a-b)t}}{-(a-b)} - \frac{1}{-(a-b)}\right]$$

$$= \frac{e^{-bt}}{a-b}\left[1 - e^{-at}e^{bt}\right]$$

$$\therefore L^{-1}\left[\frac{1}{s+a}\cdot\frac{1}{s+b}\right] = \frac{1}{a-b}\left[e^{-bt} - e^{-at}\right]$$

3. Using convolution theorem find

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right]$$

Solution:

We know that $L^{-1}[F(s)\cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 1 * \sin t$$

$$= \sin t * 1 \qquad (\because f(t)*g(t) = g(t)*f(t))$$

$$= \int_0^t \sin u \, du$$

$$= [-\cos u]_{u=0}^{u=t}$$

$$= (-\cos t) - (-1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - \cos t$$

4. Using convolution theorem find

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

Solution:

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * \frac{1}{a} L^{-1}\left[\frac{a}{(s^2+a^2)}\right]$$

$$= \cos at * \frac{1}{a}\sin at$$

$$= \frac{1}{a}(\cos at * \sin at)$$

$$= \frac{1}{a}\int_0^t \cos au \, \sin a(t-u)\,du$$

$$= \frac{1}{a}\int_0^t [\cos au \, \sin(at-au)]\,du$$

$$= \frac{1}{a}\int_0^t \left[\frac{\sin(at-au+au)+\sin(at-au-au)}{2}\right]du$$

$$= \frac{1}{2a}\int_0^t [\sin(at)+\sin a(t-2u)]\,du$$

$$= \frac{1}{2a}\int_0^t [\sin(at)+\sin a(t-2u)]\,du$$

$$= \frac{1}{2a}\left[(\sin at)u + \left(\frac{-\cos a(t-2u)}{-2a}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[(\sin at)u + \frac{\cos a(t-2u)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \left(\frac{\cos at}{2a}\right) - \left(0 + \frac{\cos at}{2a}\right)\right]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a} t \sin at$$

7.2 Tutorial Problems:

Find the inverse Laplace Transform using convolution theorem.

1. $\frac{4}{(s^2+2s+5)^2}$
2. $\frac{1}{(s^2-4)(s^2+4)}$
3. $\frac{s}{(s^2+a^2)^3}$

8 Initial and final value theorems

8.1 Initial value theorem

$$If\ \ L[f(t)] = F(s), \text{ then } \underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ sF(s)$$

8.2 Final value theorem

$$If\ \ L[f(t)] = F(s), \text{ then } \underset{t\to \infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$$

8.3 Problems based on initial value and final value theorems

Unit II-LaplaceTransform

1. If $L[f(t)] = \dfrac{1}{s(s+a)}$, find $\underset{t\to\infty}{Lt} f(t)$ and $\underset{t\to 0}{Lt} f(t)$

Solution:

We know that

$$\underset{t\to 0}{Lt} f(t) = \underset{s\to\infty}{Lt}\ sF(s)$$
$$= \underset{s\to\infty}{Lt}\ s\frac{1}{s(s+a)}$$
$$= \underset{s\to\infty}{Lt}\ \frac{1}{(s+a)}$$
$$= \frac{1}{\infty}$$
$$\therefore \underset{t\to 0}{Lt}\ f(t) = 0$$

We know that

$$\underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt}\ sF(s)$$
$$= \underset{s\to 0}{Lt}\ s\frac{1}{s(s+a)}$$
$$= \underset{s\to 0}{Lt}\ \frac{1}{(s+a)}$$
$$\therefore \underset{t\to\infty}{Lt}\ f(t) = \frac{1}{a}$$

2. Verify the initial and final value theorem for the function

$$f(t) = 1 + e^{-t}(\sin t + \cos t)$$

Solution:

Initial value theorem states that

$$\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s)$$

$$L[f(t)] = F(s) = \frac{1}{s} + L[\sin t + \cos t]_{s\to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

$$\text{L.H.S} = \underset{t\to 0}{Lt}\, f(t) = 1+1 = 2$$

$$\text{R.H.S} = \underset{s\to\infty}{Lt}\, s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right]$$

$$= \underset{s\to\infty}{Lt}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right]$$

$$= \underset{s\to\infty}{Lt}\left[1 + \frac{s^2(1+\frac{2}{s})}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}\right]$$

$$= \underset{s\to\infty}{Lt}\left[1 + \frac{(1+\frac{2}{s})}{\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}\right]$$

$$= 1+1$$

$R.H.S = 2$

$L.H.S = R.H.S$

Initial value theorem verified.

Final value theorem states that

$$\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, sF(s)$$

$$L.H.S = \underset{t\to\infty}{Lt}\left[1+e^{-t}(\sin t+\cos t)\right]$$

$$= 1+0=1$$

$$R.H.S = \underset{s\to 0}{Lt}\left[1+\frac{s(s+2)}{(s+1)^2+1}\right]$$

$$=1+0=1$$

$L.H.S = R.H.S$

Final value theorem verified.

L.H.S = R.H.S
Final value theorem verified.

8.4 Tutorial Problems:

Verify the initial and final value theorems for the functions

1. $f(t) = t^2 e^{-3t}$
2. If $F(s) = \dfrac{s^2+5s+2}{s^3+4s^2+2s}$ find $f(0)$ and $f(\infty)$
3. $f(t) = ae^{-bt}$

Unit II-LaplaceTransform

9. Problems based on solution of linear ODE of second order with constant coefficients

1. Using L.T solve $y''-3y'+2y=e^{-t}$ given y(0) –1, $y'(0)=0$

Solution:

$$y''-3y'+2y=e^{-t} \text{ and } y(0)=1,\ y'(0)=0$$

Taking L.T on bothsides,

$$L[y''(t)]-3L[y'(t)]+2L[y(t)]=L[e^{-t}]$$

$$s^2L[y(t)]-sy(0)-y'(0)-3[sL[y(t)]-y(0)]+2L[y(t)]=\frac{1}{s+1}$$

$$s^2L[y(t)]-s-0-3sL[y(t)]+3+2L[y(t)]=\frac{1}{s+1}$$

$$(s^2-3s+2)L[y(t)]=\frac{1}{s+1}+s-3$$

$$(s-1)(s-2)L[y(t)]=\frac{s^2-2s-2}{s+1}$$

$$L[y(t)]=\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}=\frac{A}{s+1}+\frac{B}{s-1}+\frac{C}{s-2}$$

$$s^2-2s-2=A(s-1)(s-2)+B(s+1)(s-2)+C(s+1)(s-1)$$

Put $s=1$, *we get*

$$1-2-2=-2B$$

$$-3=-2B$$

$$-3 = -2B$$

$$B = \frac{3}{2}$$

$$Put\ s = 2,\ we\ get$$

$$4-4-2 = 3C$$

$$C = \frac{-2}{3}$$

$$Put\ s = -1,\ we\ get$$

$$1+2-2 = 6A$$

$$A = \frac{1}{6}$$

$$L[y(\mathrm{t})] = \frac{1/6}{s+1} + \frac{3/2}{s-1} + \frac{2/3}{s-2}$$

$$= \frac{1}{6}\frac{1}{s+1} + \frac{3}{2}\frac{1}{s-1} + \frac{2}{3}\frac{1}{s-2}$$

$$y(\mathrm{t}) = \frac{1}{6}L^{-1}\left[\frac{1}{s+1}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$$

$$= \frac{1}{6}e^{-t} + \frac{3}{2}e^{t} + \frac{2}{3}e^{2t}$$

9.2 Tutorial Problems:

1. Solve : $y + \int_0^t y\,dt = t^2 + 2t$
2. Solve $(\mathrm{D}^2 + 5\mathrm{D} + 6)\mathrm{y} = 2$, given $\mathrm{y}(0) = 0,\ \mathrm{y}'(0) = 0$
3. $Solve\ \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0\ \ given\ that\ \ y = \frac{dy}{dx} = 1\ at\ x = 0\ using\ L.T.\ method$